Day-2

(*) Under sufficient smoothness of 'decay' for v_t, the solution to CE given P_0 is unique.

(*) Suppose we started with C.E & assume sufficiently nice behaved velocity field

$\Rightarrow \frac{dx_t}{dt} = v_t(x_t)$ $X_0 \sim P_0$ can be solved almost surely.

Then $X_t \sim P_t$

(*) $P_2(\mathbb{R}^d)$ = set of probability measures with $\int |x|^2 d\mu(x) < \infty$

$\mu_t : [0,T] \to P_2(\mathbb{R}^d)$ with the W_2 metric.

(*) assume $t \to \mu_t$ is a.c.

$\forall t, s \quad |\mu_t - \mu_s| \leq \int_t^s \beta(\theta)\, d\theta \quad \beta \in L^1(\mathbb{R}^+)$

Theorem ⇒ a velocity field $v_t(x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ with $v_t \in L^2(\mu_t)$ a.e. s.t $\frac{\partial P_t(x)}{\partial t} + \operatorname{div}(P_t v_t) = 0$

Stochastic Differential Equations

$$dx_t = v(t, x_t)dt + \sigma(t, x_t)\, dB_t$$

dB_t ~ Brownian Motion

$\xi_t = \frac{dB_t}{dt}$ → "white noise"

$$\mathbb{E}[\xi_t \xi_s] = \delta(t-s)$$

History:
- → Bachelier ~1900's
- → Norbert Wiener
- → Ito

Brownian Motion

It is a random function $t \to B_t(\omega) : \mathbb{R}^+ \to \mathbb{R}^d$ with

Properties
(a) $B_0 = 0$ a.s.
(b) $t \geq s$ $B_t - B_s$ independent of B_s $\forall \tau \leq s$
(c) $B_t - B_s \overset{d}{=} \mathcal{N}(0, (t-s)I)$
(d) $t \to B_t(\omega)$ is continous almost surely.

SDE

$$dx_t = v(t, x_t)dt + \sigma(t, x_t)dB_t$$

$$x_t = \int v(t, x_t)dt + \int \sigma(t, x_t)dB_t \quad \forall t \text{ a.s.}$$

Two theories
- Ito's Integral → Martingale method
- Stratonovich's Integral → mid point method

Reimann Stieltjes integral doesn't hold

$f, g \to [0, T] \to \mathbb{R}^d$

$$\int_0^T f(t)\, dg(t) = \lim_{|t_{i+1} - t_i| \to 0} \sum_{i=1}^{N} f(t_i)(g(t_{i+1}) - g(t_i))$$

→ Converges to limit when f is continous & g has finite variations. (i.e. $\sup_i |g(t_{i+1}) - g(t_i)| < \infty$)

→ It converges when f is finite variation & g is continous

$$\int_0^T f(t)\,dg(t) = f(T)g(T) - f(0)g(0) - \int_0^T g(t)\,df(t)$$

Now ⇒ $\int_0^T B_t\,dB_t$ → Problem B_t doesn't have finite variations any where.

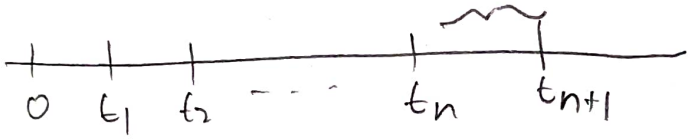
* Consider a fiteration $\mathcal{F}_t = \sigma(B_s;\ 0 \le s \le t)$

$t \to S_t$ is continous & adapted to fiteration.

$$\int_0^T S_t\,dB_t = \lim_{\sup_n |t_n - t_{n-1}| \to 0} \sum_{i=1}^{I} S_{t_i}\,(B_{t_{i+1}} - B_{t_i})$$

Indepent of B_{t_i}

(*) This convergeneare in $L_2^2(\Omega)$



→ this limit is the Ito integral

Existence & Uniqueness of solution

→ $v(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$.

$\sigma(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d \times d'}$

Setting (A) σ, v are uniformly lipschitz in space & uniformly continous in time.

Strong Solutions given a Brownian motion

(*) $\exists\ t \to X_t$ s.t. : $X_t = \int v(t, X_t)\,dt + \int \sigma(t, X_t)\,dB_t$

(*) X_t is measurable w.r.t $\sigma(B_s,\ 0 \le s \le t)$

(*) $X_0 = x_0$

Theorem Under setting (A) $\exists$ a unique storg solution. is also continous

(*) Suppose we solved $X_t = \int_0^t v(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s$

Let $\phi \in C_c^\infty([0,T] \times \mathbb{R}^d)$

$$\phi(t+\varepsilon, X_{t+\varepsilon}) - \phi(t, X_t)$$

$$= \frac{\partial}{\partial t}\phi(t, X_t)\varepsilon + \langle \nabla\phi(t, X_t), X_{t+\varepsilon} - X_t \rangle + O(\varepsilon^2)$$

By Ito integral,

$$X_{t+\varepsilon} - X_t \approx v(t, X_t)\varepsilon + \sigma(t, X_t)(B_{t+\varepsilon} - B_t)$$

[X not true. as SDE increase $\sqrt{\varepsilon}$ 2nd order matter]

$$= \frac{\partial}{\partial t}\phi(t, X_t)\varepsilon + \varepsilon\langle \nabla\phi, v(t, X_t)\rangle$$

$$+ \langle \nabla\phi(t, X_t), \sigma(t, X_t)(B_{t+\varepsilon} - B_t)\rangle$$

$$+ \frac{1}{2}(B_{t+\varepsilon} - B_t)^T \sigma^T \nabla^2\phi\, \sigma (B_{t+\varepsilon} - B_t) + O(\varepsilon)$$

$$\phi(T, X_T) - \phi(0, X_0) = \sum_{i=1}^{T} \phi(t_i, X_{t_i}) - \phi(t_{i-1}, X_{t_{i-1}})$$



$$\approx \varepsilon \sum_{i=1}^{N-1} \frac{\partial\phi(t, X_{t_i})}{\partial t} + \varepsilon\Big(\nabla\phi_i\, v(t_i, X_{t_i})$$

$$+ \langle \nabla\phi_i, B_{t_i+t} - B_t\rangle$$

$$+ \sum_i (B_{t_{i+1}} - B_{t_i})^T \sigma^T \nabla^2\phi_i\, \sigma(B_{t_{i+1}} - B_{t_i})$$

$$= \int \frac{\partial\phi}{\partial t}(t, X_t) + \int$$

$$+ \int \langle \nabla\phi(t, X_t), dB_t\rangle$$

$$+ \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{i=1}^{N-1} (B_{t_{i+1}} - B_{t_i})\, \sigma_i^T \nabla^2\phi_i\, \sigma_i (B_{t_{i+1}} - B_{t_i})$$

$$\hookrightarrow = \frac{1}{2}\int \mathrm{tr}(\sigma\sigma^T \nabla^2\phi)(t, X_t)\,dt$$

$\phi(T, X_T) - \phi(0, X_0)$ (Ito's Formula)

$$= \int_0^T \frac{\partial \phi(t, X_t)}{\partial t} dt + \int_0^T \langle \nabla \phi(t, X_t), v(t, X_t) \rangle dt$$

$$+ \int_0^T \langle \nabla \phi(t, X_t), \sigma(t, X_t) dB_t \rangle$$

$$+ \frac{1}{2} \int_0^T \mathrm{Tr}\left(\sigma \sigma^T \nabla(\phi)\right)(t, X_t)\, dt$$

$\forall\ \phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ (Fokker-Plank equation)

$$0 = \int dt \int_{\mathbb{R}^d} dP_t(x) \left[\frac{d\phi(t,x)}{\partial t} + \langle \nabla \phi(t,x), v(t,x) \rangle + \frac{1}{2} \mathrm{Tr}(\sigma \sigma^T \nabla^2 \phi)(t,x) \right]$$

$$\Rightarrow 0 = \int \phi(t,x)\, dt\, dx \left[\frac{\partial}{\partial t} P_t(x) + \mathrm{div}_x(P_t(x) v(t,x)) - \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij} P_t(x)) \right]$$

$[D = \sigma \sigma^T]$

Ex.1 $dX_t = dB_t$ $\quad v(t,x) = 0$ $\quad \sigma(t,x) = I$

Heat Equation $\left\{ \frac{\partial}{\partial t} P_t(x) = \frac{1}{2} \Delta P_t(x) = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} P_t(x) \right.$ ← FP equation

$$P_t(x) = \int \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}} P_0(y)\, dy$$

Ornstien Uhlenbeck Process

$$dX_t = -X_t\, dt + \sqrt{2}\, dB_t$$

FPeq $\quad \frac{\partial P_t(x)}{\partial t} + \mathrm{div}(-x P_t(x)) = \Delta P_t(x)$

Take $\phi(t,x) = e^t x$ & apply Ito's formula.