Brief Summary of Ito Calculus

1. B. Oksendal, Stochastic Differential Equations, Springer 1998.

We have seen SDEs several times now and have a reasonable understanding of the meaning of a process of the form:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \quad , \quad t \geq 0$$

$b(X_t) \to \mathbb{R}^d$, $\sigma(X_t) \to \in \mathbb{R}^{d \times d'}$, $dB_t \to \mathbb{R}^{d'}$

In this note, we formalize some of the constructions and the calculus associated with this process. In the integral form, the above equation is:

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \underbrace{\int_0^t \sigma(X_s)\,dB_s}_{\text{Ito Integral.}}$$

The prob. space we use is denoted by $(\Omega, \mathcal{F}, P)$.

Recall that the Brownian motion is an independent increment process, with $B_0 = 0$, and

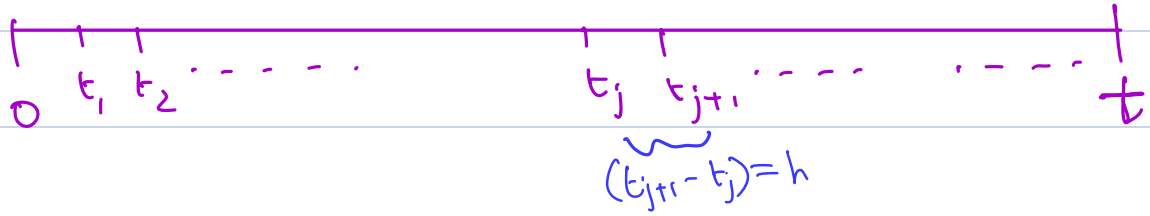
$$(B_{t+h} - B_t) \sim N(0, h).$$

The term: $\int_0^t b(X_s)\,ds$ is clear, and is defined through the usual Reimann sum limit.

$\int_0^t \sigma(X_s)\,dB_s$ is an Ito Integral. Both $\sigma(X_s)(\omega)$ and $B_s(\omega)$ are random processes.

For a "simple" random process of the form

$$f_t(\omega) = \sum_{j=1}^{n} e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t), \quad \begin{matrix} t_0 = 0 \\ t_n = t \end{matrix}$$



we define

$$\int_0^T f_t(\omega)\, dB_s(\omega) = \sum_j e_j(\omega) \cdot \left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega) \right)$$

In general, we can approximate any "reasonable" function f by simple functions over increasing finer discretization of time, and take a limit in the mean-square sense ($h \to 0$). Details can be found in [1].

Furthermore, using such a discretization, using B.M. properties described above, we have **Ito Isometry**:

$$E\left[\left(\int_s^t f_s(\omega)\, dB_s(\omega) \right)^2 \right] = E\left[\int_s^t f_s^2(\omega)\, dt \right]$$

We also state some of the standard properties of the Ito Integral (these are proved using elementary functions, and taking the limit in $L^2(P)$).

mean square limit

1. $$\int_r^t f_s(\omega)\, dB_s(\omega) = \int_r^u f_s(\omega)\, dB_s(\omega) + \int_u^t f_s(\omega)\, dB_s(\omega)$$

2. $$\int_r^t \left(c f_s(\omega) + g_s(\omega)\right) dB_s(\omega) = c \int_r^t f_s(\omega)\, dB_s(\omega) + \int_r^t g_s(\omega)\, dB_s(\omega)$$

3. $$E\left[\int_r^t f_s(\omega)\, dB_s(\omega)\right] = 0$$

4. The Ito integral is a **Martingale**

Definition: (**Filtration**) over $(\Omega, \mathcal{F}, \mathbb{P})$

$\{\mathcal{F}_t, t \geq 0\}$ is a family of σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}$, s.t. $\forall\, s < t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$

(i.e., an "increasing" sequence of σ-algebras).

Martingale: $\{M_t\}$ is a martingale wrt to $\{\mathcal{F}_t\}$ if

(i) M_t is $\mathcal{F}_t$-measurable,

(ii) $E[|M_t|] < \infty$

(iii) $E[M_t \mid \mathcal{F}_s] = M_s$ a.s. $\forall\, s \le t$.

Let $M_t(\omega) = \int_0^t f_s(\omega)\, dB_s(\omega)$, $\mathcal{F}_t = \sigma\{B_s, s \le t\}$

Ito Integral

Then, $M_t(\omega)$ is a martingale w.r.t $\mathcal{F}_t$, and

$$P\left(\sup_{0 \le t \le T} |M_t| > a\right) \le \frac{1}{a^2} E\left[\int_0^T f_s^2(\omega)\, ds\right]$$

(proof uses Doob's maximal inequality)

All of above naturally generalizes to d-dimensional B.M; see [1] for details.

The Ito Formula

(scalar case)

$$dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dB_t.$$

$Y_t = g_t(X_t)$, $g_t(x)$ is twice cont. differtiable over $t \times x \in [0, \infty) \times \mathbb{R}$

Then,

$$dY_t = \frac{\partial}{\partial t} g_t(X_t)\,dt + \frac{\partial}{\partial x} g_t(X_t)\,dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} g_t(X_t) \cdot (dX_t)^2$$

where

$(dX_t)^2$ computed using these rules:

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$$

$$dB_t \cdot dB_t = dt.$$

The proof of this follows by approximating dt and dB_t by discrete differences $(\Delta t,\ \Delta B_t = B_{t+h} - B_t)$

and letting the approximation time-step h go to '0'. The limit is taken in the $L_2(P)$ sense, and formalizes the calculation we did when we first studied SDEs in (Note 3). The key ideas are to use Taylor's Theorem, and that $(\Delta t)^2 = o(\Delta t)$, $E[(\Delta B_t)^2] \approx \Delta t$, and $E[\Delta B_t] = 0$.

In the vector case, we have the analogous result: $X_t, Y_t \in \mathbb{R}^d$, $g_{t,k} : \mathbb{R}^d \longrightarrow \mathbb{R}$,

$$Y_t = g_t(X_t) \qquad g_t(\cdot) = \begin{pmatrix} g_{t,1}(\cdot) \\ \vdots \\ g_{t,d}(\cdot) \end{pmatrix}$$

$$dY_k = \frac{\partial g_{t,k}(X_t)}{\partial t} + \sum_i \frac{\partial}{\partial x_i} g_{t,k}(X_t)\, dX_t$$

$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} g_{t,k}(X_t)\, (dX_{t,i} \cdot dX_{t,j})$$

where

$$dB_{t,i} \cdot dB_{t,j} = \delta_{ij}\, dt, \qquad dB_{t,i} \cdot dt = 0,$$

$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$ etc.

Existence / Uniqueness of Solution to SDE:

$$dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dB_t$$

Suppose $|b_t(x)| + |\sigma_t(x)| \le C(1+|x|)$, $t \in [0,T]$.

and

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \le C'|x-y|,$$

then the SDE has a unique t-continuous (i.e., cont. sample path) solution.

Strong Markov Property

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t.$$

$f(\cdot)$ a bounded fn; τ a stopping time w.r.t. $\{\mathcal{F}_t\}$.

Then, $E_{x_0=x}\left[f(X_{\tau+h}) \middle| \mathcal{F}_\tau \right] =$

$$E_{x_0=x_\tau}\left[f(X_h) \right]$$

We have already seen the **Generator** $\mathcal{A}$:

$$\mathcal{A}f(x) = \lim_{t \downarrow 0} \frac{E_{x_0=x}[f(X_t)] - f(x)}{t}, \quad x \in \mathbb{R}^n$$

For: $dX_t = b(X_t)dt + \sigma(X_t)dB_t$,

recall:

$$\mathcal{A}f(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Dynkin's Formula (aka fundamental Thm of calculus for Ito processes)

Let f a square-integrable fn, τ a stopping time

w.r.t. $\mathcal{F}_t$,

$$E_{x_0=x}[f(X_\tau)] = f(x) + E_{x_0=x}\left[\int_0^\tau \mathcal{A} f(X_s)\, ds\right]$$

Girsanov's Theorem

We next show that if we have two SDEs that are driven by the same Brownian motion (i.e., a coupling) that have different drift terms, then their distributions are "close" to each other.

Before we study this, we first have the following result:

Theorem 1 (Thm 8.6.1 in [1]) **Levy's Characterization of Brownian Motion**

Let $X_t = (X_{t,1} \cdots X_{t,d})^T \in \mathbb{R}^d$ be

a d-dim. continuous stochastic process over $(\Omega, \mathcal{F}, P)$.

Then, (a) $\Leftrightarrow$ (b):

(a) X_t is a Brownian motion w.r.t P

$\Updownarrow$

(b) (i) X_t is a martingale w.r.t. $(\{\mathcal{F}_t\}, Q)$

where $\mathcal{F}_t = \sigma\{X_s, s \le t\}$

(ii) $(X_{t,i} \cdot X_{t,j}) - \delta_{ij} t$ is a martingale w.r.t

$(\{\mathcal{F}_t\}, Q)$.

Lemma 1 (8.6.2 in [1]). ν, μ s.t. $\frac{d\nu}{d\mu} = f$.

X an absolutely integrable r.v over $(\Omega, \mathcal{F}, P)$. $\mathcal{H} \subseteq \mathcal{F}$:

Then, $E_\mu[F \cdot X | \mathcal{H}] = E_\nu[X|\mathcal{H}] \, E_\mu[F|\mathcal{H}]$.

Theorem 2: (Girsanov's Theorem) (Thm 8.6.4 in [6])

Let $Y_t \in \mathbb{R}^d$ be a Ito process of the form:

$$dY_t = \beta_t(Y_t)\,dt + \sigma_t(Y_t)\,dB_t, \quad t \in [0,T]$$

$B_t \in \mathbb{R}^{d'}$ is a Brownian motion, over $(\Omega, \mathcal{F}, P_T)$,
$\beta_t(\cdot): \mathbb{R}^d \to \mathbb{R}^d$, and $\sigma_t(\cdot): \mathbb{R}^d \to \mathbb{R}^{d \times d'}$ (a $d \times d'$ matrix).

Suppose that $\exists\ u_t(\cdot)$, $\alpha_t(\cdot)$ (both $\mathbb{R}^d \to \mathbb{R}^d$)

s.t. $\sigma_t(Y_t)\,u_t(Y_t) = \beta_t(Y_t) - \alpha_t(Y_t)$,

and

$$E_{P_T}\left[e^{\frac{1}{2}\int_0^T u_s^2(Y_s)\,ds}\right] < \infty \qquad \text{(Novikov's condition)}$$

Let $M_t = e^{\left(-\int_0^t u_s(Y_s)\,dB_s - \frac{1}{2}\int_0^t u_s^2(Y_s)\,ds\right)}$

$\int_0^t u_s(Y_s)\,dB_s = \sum_{i=1}^{d} \int_0^t u_{s,i}(Y_s)\,dB_{s,i}$

$0 \le t \le T$,

and a new measure Q_T that is absolutely

continuous w.r.t P_T, with M_t being the Radon-Nikodym derivative, i.e.,

$$dQ_T(\omega) = M_T(\omega)\, dP_T(\omega) \quad \text{over} \quad \mathcal{F}.$$

Then, we have:

(A) $\widehat{B}_t = \int_0^t u_s(Y_s)\,ds + B_t$ is a B.M. over $(\Omega, \mathcal{F}, Q_T)$

(B) The process Y_t has the Stochastic Integral representation: $dY_t = a_t(Y_t)\,dt + \sigma_t(Y_t)\,d\widehat{B}_t$

Proof: We prove in two parts.

Step (A): We show that

$$\widehat{B}_t = \int_0^t u_s(Y_s)\,ds + B_t \quad \text{is a B.M. over } (\Omega, \mathcal{F}, Q_T)$$

i.e., $d\hat{B}_t = u_t(Y_t)\,dt + dB_t$

Here, to avoid technicalities, we assume $u_t(\cdot)$ is bounded. To show $\hat{B}_t$ is a B.M over $(\Omega, \mathcal{F}, Q_T)$, we Levy's char. of B.M. and verify Theorem 1, b(i) and b(ii).

b(i): $\hat{B}_t$ is a martingale w.r.t. Q_T:

Let $K_t = M_t \cdot \hat{B}_t$, $M_t \in \mathbb{R}$, $\hat{B}_t \in \mathbb{R}^d$, $K_t \in \mathbb{R}^d$.

i.e., $K_{t,i} = M_t \cdot \hat{B}_{t,i}$, $i = 1, 2, \ldots, d$.

where

$$M_t = e^{\left(-\int_0^t u_s(Y_s)\,dB_s - \frac{1}{2}\int_0^t u_s^2(Y_s)\,ds\right)}$$

($u\,dB \equiv \langle u, dB\rangle$, $u^2 = \langle u, u\rangle$)

Applications of Ito's Formula: (problems 4.3, 4.4 in [1])

(a) $d(X_t Y_t) = X_t\,dY_t + Y_t\,dX_t + dX_t \cdot dY_t$.

(b) $Z_t = e^{-\int_0^t \theta_s(z_s)\,dB_s - \frac{1}{2}\int_0^t \theta_s^2(z_s)\,ds}$

Then $dZ_t = -Z_t \cdot \theta_t(z_t)\,dB_t$, $\quad z_t \in \mathbb{R}$

$\theta_t(\cdot) \in \mathbb{R}^d$,

$\theta_t(\cdot)\,dB_t \triangleq \sum_{i=1}^{d} \theta_{t,i}(\cdot)\,dB_{t,i}$ $\quad B_t \in \mathbb{R}^d$

$\therefore\ dK_{t,i} \overset{(a')}{=} M_t\,d\hat{B}_{t,i} + \hat{B}_{t,i}\,dM_t + d\hat{B}_{t,i}\cdot dM_t$

$d\hat{B}_{t,i} = u_{t,i}(Y_t)\,dt + dB_{t,i}$

$dM_t = -M_t \sum_{k=1}^{d} u_{t,k}(Y_t)\,dB_{t,k}$ $\quad$ from (b')

$= M_t\left(u_{t,i}(Y_t)\,dt + dB_{t,i}\right)$

$+ \hat{B}_{t,i}\left(-M_t \sum_{k=1}^{d} u_{t,k}(Y_t)\,dB_{t,k}\right)$

$+ \left(u_{t,i}(Y_t)\,dt + dB_{t,i}\right)\left(-M_t \sum_{k=1}^{d} u_{t,k}(Y_t)\,dB_{t,k}\right)$.

$dt \cdot dB_{t,i} = 0$

$dB_{t,i}\cdot dB_{t,j} = \delta_{ij}$

$= M_t\left(dB_{t,i} - \hat{B}_{t,i} \sum_{k=1}^{d} u_{t,k}(Y_t)\,dB_{t,k}\right)$

$= M_t\left(\gamma_t^{(i)} \cdot dB_t\right)$, $\quad \left(\gamma_t^{(i)} \in \mathbb{R}^d,\ B_t \in \mathbb{R}^d\right)$

$M_t \in \mathbb{R}$.

where $\gamma^{(i)}_{t,j} = \begin{cases} -\hat{B}_{t,i}\, u_{t,j}(\gamma_t) & j \neq i \\ 1 - \hat{B}_{t,i}\, u_{t,i}(\gamma_t) & j = i \end{cases}$

i.e., $K_{t,i} = \int_0^t M_s \left(\gamma^{(i)}_s \cdot dB_s\right)$ is an Ito Integral,

and thus $K_{t,i}$ is a martingale w.r.t. P.

$E_\nu[f \cdot X | \mathcal{H}] = E_\nu[X|\mathcal{H}]\, E_\mu[f|\mathcal{H}]$, $f = \frac{d\nu}{d\mu}$

∴ From Lemma 1, $(\nu \to Q, \mu \to P, f \to M_t, X \to \hat{B}_t)$

$$E_Q\left[\hat{B}_{t,i} \,\middle|\, \mathcal{F}_s\right] = \frac{E_P\left[M_t \hat{B}_{t,i} \,\middle|\, \mathcal{F}_s\right]}{E_P\left[M_t \,|\, \mathcal{F}_s\right]}$$

where $\mathcal{F}_t = \sigma\{B_s,\ s \le t\} = \sigma\{\hat{B}_s,\ s \le t\}$.

But $E_P\left[M_t \,|\, \mathcal{F}_s\right] = M_s$ (M_t is a martingale, see Remark 2)

$$E_P\left[M_t \hat{B}_{t,i} \,\middle|\, \mathcal{F}_s\right] = E_P\left[K_{t,i} \,\middle|\, \mathcal{F}_s\right] = K_{s,i}$$

$$\therefore \quad E_Q\left[\hat{B}_{t,i} \,\middle|\, \mathcal{F}_s\right] = \frac{K_{s,i}}{M_s} = \hat{B}_{s,i} \,,$$

i.e., $\{\hat{B}_{t,i}\}$ is a martingale.

Similar proof for b(ii); thus from Levy's char., $\hat{B}_{t,i}$ is a B.M.

Step (B): The process Y_t has the Stochastic Integral representation:

$$dY_t = \alpha_t(Y_t)\,dt + \sigma_t(Y_t)\,d\hat{B}_t$$

By definition,

$$dY_t = \beta_t(Y_t)\,dt + \sigma_t(Y_t)\,dB_t$$

$$= \beta_t(Y_t)\,dt + \sigma_t(Y_t)\left(d\hat{B}_t - u_t(Y_t)\,dt\right)$$

$$= \left(\beta_t(Y_t) - \sigma_t(Y_t)\,u_t(Y_t)\right)dt + \sigma_t(Y_t)\,d\hat{B}_t$$

$$= \alpha_t(Y_t)\,dt + \sigma_t(Y_t)\,d\hat{B}_t \qquad \blacksquare$$

Remark 1: Summarizing, we have

$$dY_t = \beta_t(Y_t)\,dt + \sigma_t(Y_t)\,dB_t$$

(B_t: BM over $(\Omega, \mathcal{F}, P_T)$)

$$= \alpha_t(Y_t)\,dt + \sigma_t(Y_t)\,d\hat{B}_t$$

($\hat{B}_t$: BM over $(\Omega, \mathcal{F}, Q_T)$)

Now, let us define a new process $\tilde{Y}_t$ as:

$$d\tilde{Y}_t = \alpha_t(\tilde{Y}_t)\,dt + \sigma_t(\tilde{Y}_t)\,dB_t$$

(B_t: BM over $(\Omega, \mathcal{F}, P_T)$)

Thus, $(Y_t, \tilde{Y}_t)$ are a pair of processes that are coupled by the same B.M. B_t, but with different drift terms.

Observe that this trajectory Y_t driven by $\hat{B}_t$ (under measure Q_T) is the same in distribution as $\tilde{Y}_t$ driven by B_t (under measure P_T).

Remark 2: (From 4.4, Oksendal [1])

Suppose $\theta_t = (\theta_{t,1}, \ldots, \theta_{t,d}) \in \mathbb{R}^d$

and $M_t = e^{\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds}$, $t \in [0, T]$

where $\theta^2 = \theta \cdot \theta$.

Then (a) $dM_t = M_t (\theta_t \cdot dB_t)$

(shown using Ito's formula)

(b) Suppose $E\left[e^{\frac{1}{2}\int_0^T \theta_s^2 \, ds}\right] < \infty$ (Novikov condition).

Then M_t is a martingale.