

Characterization of $N(0,1)$

①

[1] N. Ross, Fundamentals of Stein's Method, arXiv 1109.1880, 2011.

Stein's Lemma: Define the operator A by

$$Af(x) = f'(x) - xf(x).$$

f is absolutely continuous.

(a) $Z \sim N(0,1)$

$$\Rightarrow E[Af(Z)] = 0$$

$$\forall f \text{ abs. cont. s.t. } E[f'(Z)] < \infty$$

(b) For some r.v. W , suppose that

$$E[Af(W)] = 0$$

$$\forall f \text{ abs. cont. s.t. } E[f'(Z)] < \infty$$

$$\Rightarrow W \sim N(0,1).$$

In the M.P. Note 4, we used the semi-group generator $\alpha f(x) = f''(x) - xf'(x)$.
Note that here, $\alpha = A \circ D$, where D is the differential operator.

Proof:

(a) $Z \sim N(0,1)$, f abs. cont., $E[f'(Z)] < \infty$

$$E[f'(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} f'(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-t^2/2} f'(t) dt$$

$$+ \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2} f'(t) dt$$

$$\rightarrow \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2} f'(t) dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f'(t) \int_t^{\infty} w e^{-w^2/2} dw dt$$

$$f(b) - f(a)$$

$$= \int_a^b f'(t) dt$$

fundamental thm of calculus.

(3)

$$-\frac{i}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\omega^2/2} f(\omega) d\omega =$$

$$\begin{matrix} \omega \rightarrow -\omega \\ d\omega \rightarrow -d\omega \end{matrix} \quad -\frac{i}{\sqrt{2\pi}} \int_0^{\infty} \omega e^{-\omega^2/2} f(\omega) d\omega$$

Change of
integration
order

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \omega e^{-\omega^2/2} \left(\int_0^{\omega} f'(t) dt \right) d\omega$$

|||^y,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-t^2/2} f'(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \omega e^{-\omega^2/2} \left(\int_{\omega}^0 f'(t) dt \right) d\omega$$

$f(\omega) - f(\omega)$ $f(\omega) - f(\omega)$

Putting them together,

$$= E[Z F(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega f(\omega) e^{-\omega^2/2} d\omega$$

(b). We need to show that if

$$E[F'(W)] = E[W F(W)]$$

for F abs. cont, and $E[F'(Z)] < \infty$,

then, $W \sim N(0, 1)$.

Consider the functions (absolutely cont.).

$\{f_x : x \in \mathbb{R}\}$, where

$= E[\chi_{\{Z \leq x\}}]$ (4)
 CDF of $N(0,1)$.

$$f_x(w) = e^{w^2/2} \int_w^\infty e^{-t^2/2} (\Phi(x) - \chi_{\{t \leq x\}}) dt$$

Properties:

(i) f_x is the unique bounded solution to the ODE

$$f'_x(w) - w f_x(w) = \chi_{\{w \leq x\}} - \Phi(x)$$

(ii) $\|f_x\| \leq \sqrt{\frac{\pi}{2}}$.

(Note: $\|\cdot\| \equiv \|\cdot\|_\infty$,
 i.e., $\sup_y |f_x(y)|$)

(iii) $|E[f'_x(W) - W f_x(W)]| = |P(W \leq x) - \Phi(x)|$

(i) $f'_x(w) = w f_x(w) - e^{w^2/2} \cdot e^{-w^2/2} (\Phi(x) - \chi_{\{w \leq x\}})$

chain rule

$$\Rightarrow f'_x(w) - w f_x(w) = \chi_{\{w \leq x\}} - \Phi(x)$$

(ii), Uniqueness $\rightarrow$ see [1].

(ii) follows from solving $f_x(w)$ to get

$$f_x(w) = \begin{cases} \sqrt{2\pi} e^{w^2/2} \Phi(w) (1 - \Phi(w)) & w \leq x \\ \sqrt{2\pi} e^{w^2/2} \Phi(x) (1 - \Phi(w)) & w > x. \end{cases}$$

(iii) follows from applying (i) to the r.v. W , and taking expectations.

∴ For f_x , we have from (iii), and (b),

$$\begin{aligned} 0 &= E [f'_x(w) - w f_x(w)] \\ &= |P(W \leq x) - \Phi(x)| \end{aligned}$$

$$\forall x \in \mathbb{R}$$

$$\Rightarrow W \sim N(0, 1).$$

